

Decision trees, monotone functions, and semi-matroids

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Abstract

We define decision trees for monotone functions on a simplicial complex. We generalize the definition of semi-nonevasiveness to monotone functions, and show that various monotone functions related to semi-matroids are semi-nonevasive. The motivating example is the complex of bipartite graphs, whose Betti numbers are unknown in general. We show that these monotone functions have optimum decision trees, from which we can compute relative Betti numbers of related pairs of simplicial complexes. Moreover, these relative Betti numbers are coefficients of evaluations of the Tutte polynomial, and every semi-matroid collapses onto its broken circuit complex.

1 Introduction

Let $f : 2^S \rightarrow \{0, 1\}$ be a function. We wish to find a decision tree for computing f , and to minimize the depth of this decision tree, which is always at most $|S|$. In the current paper, we consider decision trees for functions $f : \Delta \rightarrow P$, for some simplicial complex Δ and some set P . We extend the notion of evasiveness to this setting. We also consider the problem of finding an *optimal* decision tree when P is a poset, and f is monotone-increasing. In this case, an *optimal* decision tree gives information about relative homology groups $\tilde{H}_i(\Delta_p, \Gamma_p)$, where Δ_p are all the faces whose function value is at most p , and Γ_p consists of all faces whose function value is strictly less than p .

There are numerous papers involving cases where f is a monotone graph property, such as being a disconnected graph [Vas90], or being bipartite [Cha]. Given such a function f , let $\Delta = f^{-1}(0)$, which is an abstract simplicial complex. A well-known fact is that nonevasive complexes are contractible. So several graph properties have been proved to be evasive by showing that Δ has nontrivial homology in some dimension. We refer the interested reader to the book *Simplicial complexes of Graphs* [Jon08b], Jonsson, which provides an extensive survey of several such complexes, and what is known about their topology. When the graph property is the property of being acyclic [PB80], bipartite [Cha], or not connected [Vas90], the complex is actually known to be homotopy equivalent to a wedge of spheres.

Also, Robin Forman [For95] noted that any decision tree can be used to give inequalities on the Betti numbers of Δ . When the inequalities are actually equalities, then we

obtain the notion of semi-nonevasiveness, due to Jonsson [Jon05]. Jonsson showed that the properties of being bipartite, disconnected, or acyclic are all semi-nonevasive [Jon08b]. Moreover, he showed that this same result holds for a large family of simplicial complexes which he calls *pseudo-independence complexes*. However, no formulas are given for the Betti numbers at this level of generality, and computing these Betti numbers was the primary motivation for this paper.

We show that the definition of strong pseudo-independence complexes is actually a new definition of semi-matroids, which is why we concern ourselves with semi-matroids. Semi-matroids were originally introduced by Ardila [Ard07], who showed that they come with a Tutte polynomial which has nonnegative coefficients. We show that the only non-zero Betti number of a semi-matroid is actually the constant term of the Tutte polynomial. For hyperplane arrangements, it is known that this constant term also gives the only nonzero Betti number of the broken circuit complex. We discovered that every semi-matroid actually collapses onto its broken circuit complex, and moreover, this is done in a particularly nice way; faces are collapsed onto faces of the same rank. It was trying to understand this collapsing that led us to consider semi-nonevasiveness of monotone functions. We show that several functions on semi-matroids, including the rank function and nullity function, are semi-nonevasive. We show that the relative Betti numbers of these functions can be computed from the Tutte polynomial.

The paper is organized as follows: first, we review the definition of semi-nonevasive for simplicial complexes, and extend it to *convex set families*. In Section 3, we study decision trees and semi-nonevasiveness for monotone functions. We present this material first as it may be of independent interest, and is not directly related to our study of semi-matroids. In Section 4, we review semi-matroids, and prove that various functions on semi-matroids are semi-nonevasive. In Section 5, we relate our results to the study of broken circuits. In Section 6, we show the equivalence between strong pseudo-independence complexes and semi-matroids. In Section 7, we conclude with some open problems.

2 Semi-nonevasiveness

We give a brief overview of decision trees. Informally speaking, we are given a family $\mathcal{S} \subseteq 2^S$, and a mystery set $\sigma \subseteq S$. Our goal is to determine whether or not $\sigma \in \mathcal{S}$ by asking questions of the form “is $x \in \sigma$ ” for $x \in S$. We want to minimize the number of questions asked. The resulting questions can be encoded as a decision tree.

A \mathcal{S} -tree over S is a full binary tree, whose internal nodes are labeled with elements of S , and whose leaves are labeled Y and N (for yes and no, respectively). Let x be the root of T , T_1 be the left subtree, and T_2 be the right subtree. Given σ , we query if $x \in \sigma$. If it is, we recurse to T_1 ; otherwise, we recurse to T_2 . Upon reaching a leaf, we return its label. This defines a function $T : \mathcal{S} \rightarrow \{Y, N\}$, and the *decision algorithm*.

A \mathcal{S} -tree T is a *decision tree* if, given $\sigma \subseteq S$, we have $T(\sigma) = Y$ if and only if $\sigma \in \mathcal{S}$. A set $\sigma \in \mathcal{S}$ is *evasive* (with respect to T) if the decision algorithm asks $|S|$ questions before we reach a leaf node. If there are no evasive sets, then \mathcal{S} is *nonevasive*.

One commonly studied case is when \mathcal{S} is a simplicial complex. One of the purposes of this paper is to study convex families as well.

Definition 2.1. *An abstract simplicial complex Δ over S is a collection of subsets of S such that, if $\sigma \in \Delta$ and $\tau \subseteq \sigma$, then $\tau \in \Delta$. A convex family is a collection of sets \mathcal{S} over S , such that $\tau \in \mathcal{S}$ whenever there exists $\sigma, \rho \in \mathcal{S}$ such that $\sigma \subset \tau \subset \rho$.*

Note that it is common for authors to assume that $\{x\} \in \Delta$ for all $x \in S$. We do not make this assumption.

Given a set family \mathcal{S} over S , and $x \in S$, define $\text{del}_x(\mathcal{S}) = \{\sigma \in \mathcal{S} : x \notin \sigma\}$, and $\text{lk}_x(\mathcal{S}) = \{\sigma \subseteq S - x : \sigma + x \in \mathcal{S}\}$. Naturally, the deletion and link of a convex family is a convex family. One can show that nonevasiveness can also be defined recursively, using the notion of link and join.

Definition 2.2. *A convex family \mathcal{S} is nonevasive if:*

1. $\mathcal{S} = \emptyset$,
2. \mathcal{S} is the 0-simplex,
3. *there exists a vertex x (called a shedding vertex) such that $\text{del}_x(\mathcal{S})$ and $\text{lk}_x(\mathcal{S})$ are both nonevasive.*

In the case \mathcal{S} is a simplicial complex, we recover the usual notion of nonevasiveness. It is also known that there is a relationship between decision trees and discrete Morse theory. It turns out this relationship can be generalized to convex families as well, via relative homology. That is, given a convex family \mathcal{S} , we define $\Gamma = \{\sigma : \sigma \subseteq \tau \text{ for some } \tau \in \mathcal{S}\}$, and $\Delta = \Gamma \setminus \mathcal{S}$. Then $\Delta \subseteq \Gamma$ and both are simplicial complexes. (Δ, Γ) is the *simplicial pair* associated to \mathcal{S} .

There are many ways of defining relative homology groups associated to a simplicial pair (Δ, Γ) . For our purposes, the easiest way to define it is by $\tilde{H}_d(\Gamma/\Delta) \simeq \tilde{H}_d(\Gamma \cup \text{cone}(\Delta))$. To simplify notation, we define $\tilde{H}_d(\mathcal{S}) = \tilde{H}_d(\Gamma/\Delta)$. The *Poincaré* polynomial is $P(\mathcal{S}; q) = \sum_{i \geq -1} \tilde{\beta}_i(\mathcal{S}) q^i$, where $\tilde{\beta}_i(\Gamma/\Delta)$ is the rank of the i th relative reduced homology group.

Proposition 2.3. *Let T be a decision tree for a convex family \mathcal{S} , and let (Δ, Γ) be the associated pair. Then T induces an acyclic matching on $\Gamma \cup \text{cone}(\Delta)$, whose critical cells correspond to evasive sets of \mathcal{S} with respect to T . Moreover, $\Gamma \cup \text{cone}(\Delta)$ is homotopy-equivalent to a cell complex X , such that each i -cell corresponds to an evasive i -set.*

Proof. Suppose T is a tree with only one vertex. If the vertex is labeled Y , then it must be the case that $\mathcal{S} = 2^S$, where $S \neq \emptyset$. In this case, there are many possible acyclic matchings, and we are done. If the vertex is labeled N , then $\mathcal{S} = \emptyset$, and hence $\Gamma \cup \text{cone}(\Delta) = \text{cone}(\Delta)$. A perfect matching is given by matching any set $\sigma \in \Delta$ with $\sigma + x$, where x is the new cone point.

Finally, suppose that T has more than one vertex, let x be the label of the root, and let T_l, T_r be the left and right subtrees. Then by induction T_l induces an acyclic matching

M_l on $\text{del}_x(\Gamma \cup \text{cone}(\Delta))$, and T_r induces an acyclic matching M_r on $\text{lk}_x(\Gamma \cup \text{cone}(\Delta))$. Define $M'_r = \{(\sigma + x, \tau + x) : (\sigma, \tau) \in M_r\}$. Then $M_l \cup M'_r$ is an acyclic matching on $\Gamma \cup \text{cone}(\Delta)$, and the result follows. \square

Consider a decision tree T , and define $ev_T(\mathcal{S}; q) = \sum q^{\dim \sigma}$ where the sum is over all evasive sets of \mathcal{S} . Then, as a result of our proposition, we have $[q^i]ev_T(\mathcal{S}; q) \geq \tilde{\beta}_i(\mathcal{S})$. When equality holds for all i , then T is an *optimal* decision tree. In other words, the relative Betti numbers correspond to the evasive faces of dimension i .

Definition 2.4. *Let \mathcal{S} be a convex family. Then \mathcal{S} is semi-nonevasive if:*

1. $\mathcal{S} = \emptyset$,
2. \mathcal{S} is the 0-simplex or (-1) -simplex,
3. *there exists a vertex x (called a shedding vertex) such that $\text{del}_x(\mathcal{S})$ and $\text{lk}_x(\mathcal{S})$ are both semi-nonevasive, and*

$$\tilde{H}_d(\mathcal{S}) \simeq \tilde{H}_d(\text{del}_x(\mathcal{S})) \oplus \tilde{H}_{d-1}(\text{lk}_x(\mathcal{S}))$$

for all d .

A simple induction argument can be used to prove the following theorem.

Theorem 2.5. *A convex family \mathcal{S} is semi-nonevasive if and only if there exists an optimal decision tree for \mathcal{S} . \mathcal{S} is nonevasive if and only if the optimal decision tree has no evasive faces.*

We end with a lemma which is useful for proving semi-nonevasiveness in cases where the evasive faces are all equidimensional.

Lemma 2.6. *Let \mathcal{S} be a convex family, and suppose there exists a decision tree T such that $ev_T(\mathcal{S}; q) = ct^{d-1}$ for some nonnegative integers c and d . Then \mathcal{S} is semi-nonevasive, and*

$$\Gamma \cup \text{cone}(\Delta) \simeq \bigvee_c \mathbb{S}^{d-1}$$

3 Nonevasive and Semi-nonevasiveness for monotone functions

Throughout this section, let P be a fixed poset, and let Δ be a simplicial complex. A function $f : \Delta \rightarrow P$ is *monotone* if $f(\sigma) \leq f(\tau)$ whenever $\sigma \subseteq \tau \in \Delta$. Given a poset P , let \hat{P} be obtained from P by adding a new maximum element $\hat{1}$. Also, extend f to a function $\hat{f} : 2^S \rightarrow \hat{P}$ by setting $\hat{f}(\sigma) = \hat{1}$ for every $\sigma \notin \Delta$.

Let T be a full binary tree, whose internal nodes are labeled by elements of S , and whose leaves are labeled by elements of \hat{P} . Let x be the root of the tree. Given a mystery

set σ , we query whether or not $x \in \sigma$, and recurse to the appropriate subtree. When we reach a leaf, we return the value of the leaf. This defines a function $T : 2^S \rightarrow \hat{P}$. The tree T is a *decision tree* for f if and only if $T(\sigma) = \hat{f}(\sigma)$ for all $\sigma \subseteq S$.

Given a decision tree T , a set σ is *evasive* if the decision algorithm for σ has to ask $|\sigma|$ questions before returning $T(\sigma)$. Given a decision tree T , and $p \in P$, let $ev_{T,p}(f; q) = \sum q^{\dim \sigma}$ where the sum is over all evasive sets σ of T with $f(\sigma) = p < \hat{1}$. If there are no evasive sets, we say f is *nonevasive*. It turns out that nonevasiveness also has a recursive definition. Given $x \in S$, we define $f_{\setminus x} = f|_{\text{del}_x \Delta}$ and $f_{/x} : \text{lk}_x(\Delta) \rightarrow P$ by $f_{/x}(\sigma) = f(\sigma + x)$.

Definition 3.1. *Let $f : \Delta \rightarrow P$ be a monotone function. Then f is nonevasive if*

1. \hat{f} is constant, and $V(\Delta) \neq \emptyset$,
2. there exists a vertex x (called a *shedding vertex*) such that $f_{\setminus x}$ and $f_{/x}$ are nonevasive.

Now we relate the notion of decision tree to the notion of decision tree for a simplicial pair. Given $f : \Delta \rightarrow P$ monotone, consider $p \in P$. Suppose we have a decision tree T for f . Consider a new tree T_p , where we replace leaves labeled p with Y , and replace all other leaves with N . Then one can see that T_p is a decision tree for the convex family $f^{-1}(p)$. In particular, $ev_{T,p}(f; q) = ev_{T_p}(f^{-1}(p); q)$. Thus if we write $ev_{T,p}(f; q) = \sum c_{i,p} q^i$, then $c_{i,p} \geq \tilde{\beta}_i(f^{-1}(p))$. If we find a decision tree where equality holds for all i and p , then such a tree is *optimal*. In such a case, we have a decision tree that, for any $i \in \mathbb{N}$, and $p \in P$, minimizes the number of sets σ such that $f(\sigma) = p$ and $|\sigma| = i$.

Now we extend the notion of semi-nonevasive to monotone functions:

Definition 3.2. *Let $f : \Delta \rightarrow P$ be monotone. Then f is semi-nonevasive if:*

1. \hat{f} is constant,
2. there exists a vertex x (called a *shedding vertex*) such that $f_{\setminus x}$ and $f_{/x}$ are both semi-nonevasive, and

$$\tilde{H}_d(f^{-1}(p)) \simeq \tilde{H}_d(f_{\setminus x}^{-1}(p)) \oplus \tilde{H}_{d-1}(f_{/x}^{-1}(p))$$

for all $d \geq -1$, $p \in P$.

Lemma 3.3. *Let $f : \Delta \rightarrow P$ be monotone. Suppose there exists c_p, d_p for every $p \in \hat{P}$, such that $ev_{T,p}(f; q) = c_p q^{d_p}$. Then f is semi-nonevasive, and T is optimal.*

Proof. Consider such a decision tree T , and let $p \in P$. Then T_p , the tree given by replacing each leaf labeled p with Y , and all other leaves with N , is a decision tree for $f^{-1}(p)$. Moreover, $ev_{T_p}(f^{-1}(p); q) = c_p q^{d_p}$, so by Lemma 2.6, we see that T_p is optimal, whence $ev_{T_p}(f^{-1}(p); q) = P(f^{-1}(p); q)$. Since p was arbitrary, we see that T is optimal. \square

Finally, we end this section by discussing one attempt to define the Poincaré polynomial of a monotone function $f : \Delta \rightarrow P$. Clearly, it does not appear to be possible to do this in complete generality. However, suppose $P \subseteq \mathbb{N}^n$ for some n . Let x_1, \dots, x_n be indeterminates. Then we define

$$P(f; q, \mathbf{x}) = \sum_{p \in P} P(f^{-1}(p); q) \mathbf{x}^p$$

where $\mathbf{x}^p = x_1^{p_1} \cdots x_n^{p_n}$, and $p = (p_1, \dots, p_n)$. In some cases, we will be interested in this *Poincaré polynomial* of f .

4 Semimatroid

Semimatroids were originally introduced by Ardila [Ard07], and form a generalization of affine hyperplane arrangement (such matroids can also be called *pointed matroids*, or *affine matroids*). Ardila also studied a Tutte polynomial for semimatroids, and showed that it has nonnegative coefficients, and applied these results in a later paper involving Tutte polynomials of affine hyperplane arrangements. The goal of this section is to show that many monotone functions on a semi-matroid are semi-nonevasive, and their Poincaré polynomials are given by evaluations of the Tutte polynomial. In a later section, we shall show that semimatroids are equivalent to Jonsson's strong pseudo-independence complexes.

Definition 4.1. A semi-matroid is a triple (S, Δ, r) where Δ is a non-void simplicial complex over S , and $r : \Delta \rightarrow \mathbb{N}$ such that we have the following, for any $\sigma, \tau \in \Delta$:

1. $0 \leq r(\sigma) \leq |\sigma|$.
2. If $\sigma \subseteq \tau$, then $r(\sigma) \leq r(\tau)$.
3. If $\sigma \cup \tau \in \Delta$, then $r(\sigma) + r(\tau) \geq r(\sigma \cup \tau) + r(\sigma \cap \tau)$.
4. If $r(\sigma) = r(\sigma \cap \tau)$, then $\sigma \cup \tau \in \Delta$.
5. If $r(\sigma) < r(\tau)$, then there exists $y \in \tau \setminus \sigma$ such that $\sigma \cup \{y\} \in \Delta$.

Example 4.2. Let H be a collection of affine hyperplanes in \mathbb{R}^n . Then $\{H_1, \dots, H_k\}$ is intersecting if $\cap_{i=1}^k H_i \neq \emptyset$. Let $\Delta(H)$ be the collection of all intersecting sets of hyperplanes. Given an intersecting set σ , define $r(\sigma) = n - \dim(\cap \sigma)$, the codimension of the intersection of the hyperplanes. Then $(H, \Delta(H), r)$ is a semi-matroid.

A set $X \in \Delta$ is *dependent* if $r(X) < |X|$, and *independent* otherwise. A *circuit* is a minimal dependent set. A maximal independent set is a basis, and every basis has the same size (the rank, r_C of the semi-matroid). A circuit of size 1 is called a *loop*, and an element of S that is in every basis is called a *coloop*.

Definition 4.3. The Tutte polynomial of a semi-matroid $T_{\mathcal{C}}(x, y)$ is the polynomial

$$T_{\mathcal{C}}(x, y) = \sum_{\sigma \in \Delta} (x - 1)^{rc - r(\sigma)} (y - 1)^{|\sigma| - r(\sigma)}$$

Theorem 4.4. Let \mathcal{C} be a semi-matroid, let $e \in S$. Then $T_{\mathcal{C}}$ satisfies the following recurrence:

1. $T_{\emptyset} = 1$,
2. $T_{\mathcal{C}} = T_{\mathcal{C}-e}$ if $\{e\} \notin \Delta$,
3. $T_{\mathcal{C}} = yT_{\mathcal{C}-e}$ if e is a loop,
4. $T_{\mathcal{C}} = xT_{\mathcal{C}/e}$ if e is a coloop,
5. $T_{\mathcal{C}} = T_{\mathcal{C}-e} + T_{\mathcal{C}/e}$ otherwise.

Our main theorem is the following:

Theorem 4.5. Let $\mathcal{C} = (S, \Delta, r)$ be a semi-matroid, with $S \neq \emptyset$.

- Let $r : \Delta \rightarrow \mathbb{N}$ be the rank function. Then r is semi-nonevasive, and $P(r; q, x) = (qx)^{rc} T_{\mathcal{C}}(\frac{qx+1}{qx}, 0)$.
- Let $n : \Delta \rightarrow \mathbb{N}$ be given by $n(\sigma) = |\sigma| - r(\sigma)$. Then n is monotone and semi-nonevasive. Moreover, $P(n; q, x) = (q)^{rc} T_{\mathcal{C}}(0, qx + 1)$.
- Let $r \times n : \Delta \rightarrow \mathbb{N}$ be given by $r \times n(\sigma) = (r(\sigma), n(\sigma))$. Then $r \times n$ is semi-nonevasive. Moreover, $P(n \times r; q, x, y) = (qx)^{rc} T_{\mathcal{C}}(\frac{qx+1}{qx}, qy + 1)$.
- Δ is semi-nonevasive, and $P(\Delta; q) = T_{\mathcal{C}}(0, 0)$.

Proof. We prove the result by induction on $|S|$, where the case $|S| = 1$ can be verified directly. So suppose $|S| > 1$. Let f be r, n , or $r \times n$.

Suppose \mathcal{C} consists only of loops and coloops, in which case Δ is a full simplex. Given any coloop x , we have $\hat{n}(\sigma - x) = \hat{n}(\sigma + x)$ for any $\sigma \in \Delta$, and hence n is nonevasive if S has a coloop, as we never have to query x . Similarly, if S has a loop, then r is nonevasive. Also, if S has any loop or coloop, then Δ is nonevasive. In all remaining cases where \mathcal{C} has only loops and coloops, note that we have to query all elements of S in order to determine the value of f . In these cases, we do get a decision tree T with $ev_T(f; q, \mathbf{x}) = \sum_{\sigma \in \Delta} q^{\dim \sigma} \mathbf{x}^{f(\sigma)}$. For each case of f , the reader can verify that Lemma 3.3 applies, and hence f is semi-nonevasive.

Suppose there exists $x \in S$ such that $\{x\} \notin \Delta$. Let T be an optimal decision tree for $f_{\setminus x}$. Let T' have root x , left subtree T , and right subtree consisting of a leaf labeled $\hat{1}$. Then T' is a decision tree for f , and $ev_{T'}(f; q, \mathbf{x}) = ev_T(f_{\setminus x}; q, \mathbf{x})$, and we are done by induction.

Suppose there exists $x \in S$, and suppose that $\{x\} \in \Delta$, and x is not a loop or coloop. Then, by induction, there exists optimal decision trees $T_{\setminus x}$ for $f_{\setminus x}$, and $T_{/x}$ for $f_{/x}$. If we construct a new tree T with x as the root, and $T_{\setminus x}$, $T_{/x}$ as the subtrees, then one can see that $ev_T(f; q, \mathbf{x}) = ev_{T_{\setminus x}}(f_{\setminus x}; q, \mathbf{x}) + qtev_{T_{/x}}(f_{/x}; q, \mathbf{x})$, where $t = x$ if $f = n \times r$ or $f = r$, and $t = 1$ otherwise. Then the result follows by induction and an application of Lemma 3.3. \square

5 Broken Circuits

In this section, we review the notion of broken circuit complex of a matroid, and extend it to semi-matroids. The purpose will be to give direct combinatorial interpretations of the coefficients of Poincaré polynomials for r and Δ .

Let $\mathcal{C} = (S, \Delta, r)$ be a semi-matroid, and fix a linear order on S . A broken circuit is any face of the form $\sigma - \min \sigma$ where σ is a circuit. A non-broken circuit τ is a face which does not contain a broken circuit. Clearly, non-broken circuits form a subcomplex $BC(\mathcal{C})$ of Δ .

Theorem 5.1. *Let \mathcal{C} be a semi-matroid. Then $BC(\mathcal{C})$ is vertex-decomposable. Moreover, $f(BC(\mathcal{C}), q) = q^r T_{\mathcal{C}}(\frac{q+1}{q}, 0)$ and*

$$BC(\mathcal{C}) \simeq \bigvee_{T_{\mathcal{C}}(0,0)} \mathbb{S}^{r_{\mathcal{C}}-1}$$

Proof. Let $s = \max S$. If $\{s\} \notin \Delta$, then $BC(\mathcal{C}) = BC(\mathcal{C} - s)$, and the result follows by induction. If s is a loop, $BC(\mathcal{C}) = \emptyset$, and the result follows. So suppose s is a coloop. Then $BC(\mathcal{C})$ is the join of $\{\emptyset, \{s\}\}$ and $BC(\mathcal{C} - s)$. By induction, the latter is vertex-decomposable, and the join of vertex-decomposable complexes is vertex decomposable. Also, $BC(\mathcal{C})$ is contractible, as $BC(\mathcal{C}) = \text{cone}(BC(\mathcal{C} - s))$.

Finally, suppose s is not a loop. Then $\text{del}_s(BC(\mathcal{C})) = BC(\mathcal{C} - s)$, and $\text{lk}_s(BC(\mathcal{C})) = BC(\mathcal{C}/e)$, so vertex decomposability follows by induction. \square

A non-broken circuit σ is called *critical* if, for every $x \in \sigma$, there exists $y < x$ such that $\sigma - x + y$ is still a non-broken circuit. One can observe that $T_{\mathcal{C}}(0, 0)$ counts the number of critical non-broken circuits of size $r_{\mathcal{C}}$. This can be verified by using the linear order on S to give a shelling order on $BC(\mathcal{C})$, and then observing that critical non-broken circuits correspond to spanning simplices. Alternatively, one can verify the Tutte recursion in this case. Let $CBC(\mathcal{C})$ denote the set of critical non-broken circuits which are also bases.

Thus, we have combinatorial interpretations of the Betti numbers of \mathcal{C} , as well as the relative Betti numbers coming from $r_{\mathcal{C}}$. In fact, we can construct a decision tree for $r_{\mathcal{C}}$ whose evasive faces are the nonbroken circuits. In particular, Δ collapses onto $BC(\mathcal{C})$. We can also construct a decision tree for Δ whose evasive faces are the critical nonbroken circuits.

Theorem 5.2. *Let \mathcal{C} be a semi-matroid. Then $\Delta \searrow BC(\mathcal{C})$. Moreover, $P(r; q, t) = \sum_{\sigma \in BC(\mathcal{C})} (qt)^{|\sigma|}$, and $P(\Delta; q) = \sum_{\sigma \in CBC(\mathcal{C})} q^{|\sigma|}$.*

Proof. Fix a linear order on $S = s_1, \dots, s_n$, and consider the following decision algorithm. Let $\sigma \subseteq S$. At the i th step of the algorithm we suppose we have already queried $T \subseteq \{s_n, \dots, s_{n-i+1}\}$, and we know that $\rho = \sigma \cap T$ contains no circuits. If $\rho + s_{n-i}$ contains a circuit, then we do not query s_{n-i} , and we move on to the $(i+1)$ st step of the algorithm. Otherwise, we query s_{n-i} , and either continue on to the $(i+1)$ st step of the algorithm, or terminate when $\rho + s_{n-i} \subseteq \sigma$ and $\rho + s_{n-i} \notin \Delta$. If we continue, we note that T contains no broken circuits either.

At the $(n+1)$ st step, we have $T \subseteq \sigma$, and we know $\sigma \in \Delta$; otherwise we would have terminated at an earlier step. If we have questions that we did not ask, they correspond to elements x for which $T + x$ contains a circuit. In particular, if $T \neq \sigma$, then there are questions we did not ask, and σ is nonevasive. If $T = \sigma$, then $\sigma \in BC(\mathcal{C})$. \square

6 Strong Pseudo-Independence Complexes

Jonsson defined the notion of strong pseudo-independence complexes of matroids, to capture some of the combinatorial properties complexes like $Bip(G)$ possessed.

We can restate Jonsson's result as follows:

Theorem 6.1. *Let M be a matroid with rank function r , and assume Δ is a strong pseudo-independence complex over M . Then*

- Δ is semi-nonevasive,
- the $(r-2)$ -skeleton of Δ is vertex decomposable,
- Δ is homotopy-equivalent to a wedge of $(r-2)$ -dimensional spheres.

As we shall see in this section, strong pseudo-independence complexes are semi-matroids, so Jonsson was really giving a new definition of semi-matroid. Our work supplies at least two new extensions to Jonsson's Theorem. First, we now know that Δ actually collapses onto a vertex-decomposable subcomplex, as a result of Theorems 5.2 and 5.1. Second, we know that the number of spheres in the wedge is given by an evaluation of the Tutte polynomial. Moreover, these results rely on the semi-matroid of Δ , and not the underlying matroid M .

Let M be a matroid with ground set S , and Δ be a simplicial complex over S . Then Δ is a *pseudo-independence complex* if, whenever $\sigma \in \Delta$, $x \in E \setminus \sigma$, and $r(\sigma + x) > r(\sigma)$, then $\sigma + x \in \Delta$. We say Δ is a *strong* complex over M if, whenever $\sigma \in \Delta$, and there is $x \in S \setminus \sigma$ such that $r(\sigma + x) = r(\sigma)$, then either $\sigma + x \notin \Delta$ or x is a cone point of $\text{lk}_\sigma(\Delta)$.

Given a semi-matroid \mathcal{C} , let M be the matroid with rank function $r(\sigma) = \max\{r(\tau) : \tau \subseteq \sigma, \tau \in \mathcal{C}\}$.

Then we have the following:

Theorem 6.2. *Δ is a strong pseudo-independence complex over M .*

Proof. First we show that Δ is a pseudo-independence complex. Let $\sigma \in \Delta$, $x \in S \setminus \sigma$, and suppose $r_M(\sigma + x) > r_M(\sigma) = r_C(\sigma)$. Hence there is a set $\tau \subset \sigma$ such that $\tau + x \in \Delta$, and $r(\tau + x) = r(\sigma) + 1$. However, then there exists $t \in (\tau + x) \setminus \sigma$ with $\sigma + t \in \Delta$. Clearly the only choice for t is x , so $\sigma + x \in \Delta$, and it is a pseudo-independence complex.

Now we show that Δ is a strong pseudo-independence complex. Let $\sigma \in \Delta$, $x \in S \setminus \sigma$, and suppose $r_M(\sigma + x) = r_M(\sigma)$. If $\sigma + x \notin \Delta$, then we are done (since $x \notin \text{lk}_\sigma(\Delta)$). So suppose $\sigma + x \in \Delta$. To show x is a cone point of $\text{lk}_\sigma(\Delta)$, we must show that, for every $\sigma \subseteq \tau \in \Delta$ with $x \notin \tau$, we have $\tau + x \in \Delta$. Observe that $r((\sigma + x) \cap \tau) = r(\sigma) = r(\sigma + x)$, so we have $(\sigma + x) \cup \tau = \tau + x \in \Delta$, since Δ is a semimatroid. Thus x is a cone point, and Δ is a strong pseudo-independence complex. \square

Now we prove the converse is true:

Theorem 6.3. *Let Δ be a strong pseudo-independence complex over a matroid M with ground set S . Then $(S, \Delta, r|_\Delta)$ is a semi-matroid.*

Proof. The first three conditions in the definition of semi-matroid are automatically satisfied, since r is the rank function of a matroid. So suppose we have $\sigma, \tau \in \Delta$ with $r(\sigma) = r(\sigma \cap \tau)$. We prove $\sigma \cup \tau \in \Delta$ by induction on $\sigma \setminus \tau$. Consider $x \in \sigma \setminus \tau$, and let $\rho = \sigma \cap \tau$. Since $r(\rho) \leq r(\rho + x) \leq r(\sigma) = r(\rho)$, it follows that $x \notin \text{lk}_\rho(\Delta)$ or x is a cone point of $\text{lk}_\rho(\Delta)$. Since $\sigma \in \Delta$, the former is impossible, so we must have the latter. Since $\tau \setminus \rho \in \text{lk}_\rho(\Delta)$, we must have $\tau + x \in \Delta$. Now by applying induction to $\sigma, \tau + x$, we see that $\sigma \cup \tau \in \Delta$.

Let $\sigma, \tau \in \Delta$ such that $r(\sigma) < r(\tau)$. Since r is the rank function of a matroid, there exists $x \in \tau \setminus \sigma$ such that $r(\sigma) < r(\sigma + x)$. Since Δ is a pseudo-independence complex, it follows that $\sigma + x \in \Delta$. Thus, $(S, \Delta, r|_\Delta)$ is a semi-matroid. \square

What we have shown is that Δ is a pseudo-independence complex over M if and only if $(S, \Delta, r|_\Delta)$ satisfies conditions 1-4 in the definition of a semi-matroid, and Δ is strong over M if and only if $(S, \Delta, r|_\Delta)$ satisfies every condition for being a semimatroid except possibly condition 4.

We end this section by discussing dual strong pseudo-independence complexes. Jonsson defines a complex Δ to be SPI^* over a matroid M if and only if Δ^* , the Alexander dual of Δ , is a strong pseudo-independence complex over M^* , the dual of M . However, we see then that Δ is the Alexander dual of a semi-matroid, and Ardila showed that the Alexander dual of a semi-matroid is again a semi-matroid. So all of our results apply to SPI^* complexes as well.

7 Future Work

There are some possibilities for future work related to semi-nonevasiveness and monotone functions. Given a simplicial complex Δ , and a monotone function $f : \Delta \rightarrow \mathbb{N}$, if (Δ, f) is semi-nonevasive, it is natural to ask whether or not Δ is semi-nonevasive. However, this can be false in general. For instance, consider the graph property of being a matching.

This gives rise to the matching complex M_n . It is known that M_n does not have torsion-free homology; therefore, it is not semi-nonevasive. However, if we let $f(\sigma) = |\sigma|$, then we know that (M_n, f) is semi-nonevasive.

There are a lot of interesting graph invariants that can be studied, such as the minimum (and maximum degree) of a graph, the size of a maximum matching, the chromatic number, the connectivity of a graph. It is possible that some of these invariants are not semi-nonevasive. Let $D(G)$ denote the maximum degree of G . Then D is not semi-nonevasive, because $D^{-1}(1)$ is the matching complex. In particular, if D was semi-nonevasive, then any optimal decision tree T could be modified to give an optimal decision tree for M_n , and hence M_n is semi-nonevasive. However, this is known to be false; M_n does not have torsion-free homology (see [Jon08a] for examples of torsion). However, the minimum-degree function has not been studied much. The connectivity, chromatic number, and matching number of a graph are all interesting monotone invariants, and there are papers related to the property of being not k -connected [BBL⁺99], being t -colorable [LS03], and having a bounded matching size [LSW08].

In another direction, one could try to extend the definitions of (semi)-collapsible, vertex decomposable, shellability, and Cohen-Macaulayness to the context of $f : \Delta \rightarrow P$, or to convex set families. In the latter case, it might be possible that a given simplicial complex is not shellable, but is shellable *relative a subcomplex*. Whether or not such a property is useful awaits further inquiry.

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